the plate surface, the problem is one-dimensional and can be easily solved. The results of the present note can be used if the lamina are perpendicular to the plate surface. In general, both a and b are small for laminated composites.

References

[1]Schneider, P. J., "Conduction," *Handbook of Heat Transfer-Fundamentals*, edited by W. M. Rohsenow, J. P. Hartnett, and E. N. Ganic, 2nd ed., McGraw-Hill, New York, 1985, Chap. 4.

[2]Choo, V. K. S., *Fundamentals of Composite Materials*, Knowen, Dover, DE, 1990, Chap. 5.

Thermal Modeling of Heat Transfer

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Introduction

GYARMATI'S genuine integral principle based on the fundamentals of the thermodynamics of irreversible processes is formulated for one-dimensional unsteady heat transfer in a semi-infinite solid. The surface temperature variation is restricted to be a power function of time. The integral functional, with the help of the dual field method, is varied with respect to the thermal boundary-layer thickness as the variational parameter. The heat conduction equation is reduced to the Euler-Lagrange equation which is a simple quadratic equation solvable for the dimensionless thermal boundary-layer thickness.

Analysis

We consider a semi-infinite solid $0 \leq x < \infty$ which is initially ($t = 0$) at uniform temperature, and when $t > 0$ the boundary surface at $x = 0$ is kept at a temperature T_0 which is assumed to satisfy a power law of the form

$$T_0(t) = Mt^a \tag{1}$$

The one-dimensional heat conduction equation governing the present system is

$$\rho C_V \left(\frac{\partial T}{\partial t}\right) = \lambda \left(\frac{\partial^2 T}{\partial x^2}\right) \tag{2}$$

where ρ is the density, C_V is the heat capacity at constant volume, and λ is the thermal conductivity. The associated boundary conditions are

$$\begin{aligned} t > 0: \quad x &= 0, \quad & T &= T_0(t) \\ x &= \beta(t), \quad & T &= 0 \end{aligned} \tag{3}$$

where $\beta(t)$, the hypothetical thermal boundary-layer thickness, is a function of time.

The variational principle which describes in space and time the evolution of irreversible processes has been proposed by Gyarmati[1] in the universal form as

$$\delta \int_V (\sigma - \psi - \Phi)\, dV = 0 \tag{4}$$

where σ denotes the entropy production, and ψ and Φ are the dissipation potentials. For heat conduction in a rigid body, we write the variational principle (4) in Fourier form[1,2] as

$$\delta \int_V [-\boldsymbol{J}_q \cdot \nabla T - \lambda(\nabla T \cdot \nabla T)/2 - (\boldsymbol{J}_q \cdot \boldsymbol{J}_q)/2\lambda]\, dV = 0 \tag{5}$$

Here, V denotes a bounded region in three-dimensional space, and T the temperature. The heat flux $\boldsymbol{J}_q$ satisfies the following energy balance equation without source term

$$\rho C_V \left(\frac{\partial T}{\partial t}\right) + \nabla \cdot \boldsymbol{J}_q = 0 \tag{6}$$

The volume integral (5) is maximum at any instant of time[1] for real physical processes, and that maximum is zero. However, in the course of approximation, the volume integral generally becomes a function of time, and therefore, it is natural to integrate it over the time interval $0 < t < \infty$ during which the process has taken place. Hence, we can write the variational principle (5) as below:

$$\delta \int_0^\infty \int_V [-\boldsymbol{J}_q \cdot \nabla T - \lambda(\nabla T)^2/2 - \boldsymbol{J}_q^2/2\lambda]\, dV\, dt = 0 \tag{7}$$

In the dual field method[3] we introduce the definition of an approximate temperature field T^* from which the heat flux can be determined using the following constitutive relation:

$$\boldsymbol{J}_q = -\lambda \nabla T^* \tag{8}$$

This method prescribes that T^* satisfies all boundary and smoothing conditions which are satisfied by T. In the case of exact solution, T and T^* are equal. Substituting for $\boldsymbol{J}_q$ in the variational principle, Eq. (7), and in the energy balance, Eq. (6), we get

$$\delta \int_0^\infty \int_V [\nabla T^* \cdot \nabla T - (\nabla T)^2/2 - (\nabla T^*)^2/2]\, dV\, dt = 0 \tag{9}$$

$$\rho C_V \left(\frac{\partial T}{\partial t}\right) - \lambda \nabla^2 T^* = 0 \tag{10}$$

In the following we apply the variational principle, Eq. (9), to investigate the present problem of heat transfer in a semi-infinite solid. The variational principle, Eq. (9), equivalent to the one-dimensional heat transfer problem defined by Eq. (2), reduces to

$$\delta \int_0^\infty \int_0^\beta \left[\left(\frac{\partial T}{\partial x}\right)\left(\frac{\partial T^*}{\partial x}\right) - \left(\frac{\partial T}{\partial x}\right)^2 \Big/ 2 - \left(\frac{\partial T^*}{\partial x}\right)^2 \Big/ 2\right] dx\, dt = 0 \tag{11}$$

The energy balance, Eq. (10), for the problem considered is

$$\rho C_V \left(\frac{\partial T}{\partial t}\right) - \lambda \left(\frac{\partial^2 T^*}{\partial x^2}\right) = 0 \tag{12}$$

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Using the boundary conditions, namely

$$x = 0, \qquad T = T_0(t)$$
$$x = \beta(t), \qquad T = 0$$

together with the smoothing conditions

$$x = \beta, \qquad \left(\frac{\partial T}{\partial x}\right) = 0$$
$$x = \beta, \qquad \left(\frac{\partial^2 T}{\partial x^2}\right) = 0 \tag{13}$$

we determine the following simple polynomial function to represent the temperature distribution inside the unsteady thermal boundary layer:

$$T/T_0 = 1 - 3x/\beta + 3x^2/\beta^2 - x^3/\beta^3, \qquad (0 \le x \le \beta) \tag{14}$$

Here, β is the variational parameter to be determined by the analysis. Substitution of the temperature distribution, Eq. (14), into the balance equation, Eq. (12), integration with respect to x, and evaluation of the constant of integration using the smoothing condition $\partial T^*/\partial x = 0$ at $x = \beta$, yields

$$\frac{\partial T^*}{\partial x} = (T_0/\alpha)[a(x - 3x^2/2\beta + x^3/\beta^2 - x^4/4\beta^3 - \beta/4)/t + \beta'(3x^2/2\beta^2 - 2x^3/\beta^3 + 3x^4/4\beta^4 - \tfrac{1}{4})] \quad (0 \le x \le \beta) \tag{15}$$

where $\alpha(=\lambda/\rho C_V)$ is the thermal diffusivity, and the prime denotes the time derivative. With the help of the above flux expression and the temperature distribution, Eq. (14), the variational principle, Eq. (11), after integration with respect to x, reduces to the following form:

$$\delta \int_0^\infty \{T_0^2\beta[3a/28 + 5\beta' t/(28\beta)]/(t\alpha) - 9T_0^2/10\beta - T_0^2\beta^3[a^2/288 + \beta'^2 t^2/(112\beta^2) + a\beta' t/(96\beta)]/(t^2\alpha^2)\}\,\mathrm{d}t = 0 \tag{16}$$

The variational parameter is β, and therefore, the Euler-Lagrange equation of this variational formulation is

$$\frac{\partial L}{\partial \beta} - \left(\frac{\partial L}{\partial \beta'}\right)' = 0 \tag{17}$$

if the variational principle (16) is written in the concise form

$$\delta \int_0^\infty L\,\mathrm{d}t = 0 \tag{18}$$

using the Lagrangian density L. However, if we introduce β^*, the dimensionless thermal boundary-layer thickness, given by

$$\beta^* = \beta/\sqrt{\alpha t} \tag{19}$$

we obtain the Euler-Lagrange equation of the variational principle (16) as follows:

$$\frac{\partial L}{\partial \beta^*} = 0$$

or

$$\beta^{*4}(70a^2 + 105a + 45) - \beta^{*2}(720a + 600) - 6048 = 0 \tag{20}$$

Equation (20) is a quadratic equation in β^{*2} yielding solution for β^* for given values of the surface temperature power law time exponent a. The β^* solution thus obtained enables us to compute the temperature gradient inside the boundary layer with the help of Eq. (15). The dimensionless temperature gradient at the boundary surface is calculated with the help of the expression

$$\sqrt{(t\alpha/T_0^2)}\left(-\frac{\partial T}{\partial x}\right)_{x=0} = \beta^*(a + 0.5)/4 \tag{21}$$

It is noted that the heat transfer at the boundary surface vanishes when the surface variation temperature exponent a assumes the value -0.5; this is consistent with the results of similarity theory which gives the corresponding surface temperature decay for the insulated surface situation.

The algebraic equation, Eq. (20), and local heat transfer expression, Eq. (21), constitute a complete analytical solution to the present unsteady heat conduction problem. Therefore, it is demonstrated by the present work that this unique approximate analytical method may be employed as a tool for solving heat transfer problems.

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References

[1]Gyarmati, I., "On the 'Governing Principle of Dissipative Processes' and Its Extension to Non-Linear Problems," *Annual Physics Review*, Vol. 23, No. 7, 1969, pp. 353–378.

[2]Gyarmati, I., *Non-Equilibrium Thermodynamics, Field Theory and Variational Principles*, 1st ed., Springer-Verlag, Berlin, 1970, pp. 133–141.

[3]Stark, A., "Approximate Methods for the Solution of Heat Conduction Problems Using Gyarmati's Principle," *Annual Physics Review*, Vol. 31, No. 1, 1974, pp. 53–75.

Refractive Index Effects on Local Radiative Emission from a Rectangular Semitransparent Solid

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Introduction

LOCAL radiant emission from a two-dimensional rectangular solid at uniform temperature is analyzed when the solid refractive index n is greater than 1. Since internal emission depends on n^2, a large n can provide internal radiation much larger than blackbody radiation emitted into a vacuum. Blackbody emission from the boundary is not exceeded because of internal reflection of part of the outward-directed energy at the boundary, primarily by total reflection. Energy

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